PENCILS OF CURVES ON SMOOTH SURFACES

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[Received 24 November 1999; revised 22 September 2000]

In the study of polynomial functions of two variables over the complex numbers, it has been known from the beginning (see, for example, [**4**]) that the family was topologically trivial except at a finite number of atypical values, and an early and striking theorem [**16**] asserts that these values are characterised by a simple Euler characteristic condition. The starting point of this work was an attempt to see what became of this result for curves in finite characteristic.

To understand the geometry one is soon led to compactify the family by adding in points at infinity, and it then seems more natural to consider arbitrary pencils of curves. Since one is soon led to perform blowing up, there is little to be gained by restricting to families on a plane, so this can be replaced by an arbitrary smooth projective surface.

Moreover, to direct the study in finite characteristic it is essential to begin with a clear picture of the situation in characteristic zero. Here the paper [**20**] of Lê and Weber shows that the characterisation of atypical fibres by Euler characteristics applies locally. However their purely topological reasoning cannot be used in finite characteristic, and we need to replace it by algebraic arguments.

In finite characteristic there are two essential changes to the situation in characteristic zero. One is the existence of 'wild vanishing cycles', enumerated by the 'Swan conductor', which measures the failure of the familiar behaviour of the Euler characteristic for fibrations. The other is the failure of Bertini's theorem. Although in some sense this is now well understood, the geometrical picture is significantly changed when a failure presents itself, and we do not at present have a satisfactory account of these situations.

We begin with a preliminary section introducing our main tools and concepts. We recall a number of known results on maps from a smooth surface to a smooth curve with smooth general fibre: here the Swan conductor enters the formulae, but does not really affect the condition for a fibre to be typical. In § 3 we give a brief discussion of general pencils, and in the final section we present a characterisation of exceptional members of a pencil by Euler characteristics, assuming a 'conservative' condition equivalent to Bertini's theorem.

Let k be a field of characteristic $p \geqslant 0$. Let X be a surface over k. A point $x \in X$ is smooth if and only if the corresponding points x' over algebraically closed $k' \supset k$ are smooth. If k is algebraically closed, or more generally if k is perfect, this is equivalent to x being a regular point of X. In general, smooth implies regular but not conversely: we return to this below. A point $x \in X$ is singular if it is not smooth. If k is an algebraically closed field then X is smooth at all points if X is a non-singular surface; see, for example, [**17**].

2000 *Mathematics Subject Classification*: primary 14H20; secondary 14D05, 14E22, 14F20.

Proc. London Math. Soc. (3) 83 (2001) 257–278.

Most of this work was done while the first author was a visitor at the University of Liverpool. The kind hospitality of the Department of Mathematical Sciences at Liverpool is gratefully acknowledged. The research of the first author was partially supported by a grant 'Convenio SEUID – Royal Society' and by DGES PB97-0284-CO2.

1. *Preliminaries*

We begin by recalling a number of known results, in which the behaviour of curves in characteristic p is parallel to that in the classical case. Here and below, 'curve' means an equidimensional scheme of dimension 1 of finite type over k.

1.1. *Resolution of singularities of a curve*

Singularities of curves in finite characteristic may be resolved, as in the classical case, by blowing up points: see, for example, [**6**]. We recall some properties of this procedure, which survive in positive characteristic [**8**]. We begin with a reduced curve C lying in a smooth surface X.

A single step consists in choosing a singular point x of C and blowing X up at this point. This gives a smooth surface $\widetilde{X}$ and a map $\pi\colon \widetilde{X} \to X$. The surface $\widetilde{X}$ contains an *exceptional curve* $E = \pi^{-1}(x)$, and the *strict transform* $\widetilde{C}$ of C, which may be found as the Zariski closure of $\pi^{-1}(C) - E$. When we consider C as a divisor, we have the *total transform* $\pi^*(C)$ and the strict transform $\pi^*(C) - m_x(C)E$, which does not contain E as a component. Here $m_x(C)$ is the multiplicity of C at x. The exceptional curve E is smooth and rational, and we have $\widetilde{C}.E = m_x(C)$.

Any sequence of blowings-up yields a sequence of surfaces, which contain, as well as strict transforms of the original curve C, the strict transforms of the successive exceptional curves. In each of these surfaces, the collection of all exceptional curves lying over $x \in X$ forms a tree of curves, with normal crossings.

We distinguish the exceptional curves by suffixes, and, for example, let E_α be created by blowing up a point e_α. We continue to write E_α for the strict transforms of E_α in all higher surfaces. In the terminology of Enriques, each E_α determines an *infinitely near point* to its image $x \in X$.

The sequence of blowings-up induces a partial order on the infinitely near points e_α: we say that e_β is *proximate* to e_α if the point e_β lies on the curve E_α, and that e_β *lies above* e_α if there is a sequence of points $e_\beta, e_\gamma, \dots, e_\alpha$ with each one proximate to the next.

Let X be a smooth projective surface defined over k, let x be a closed point on X and X^* be the set of infinitely near points of x. For D an effective divisor on X and $p \in X^*$, denote by $m_p(D)$ the multiplicity of the strict transform of D at p. If E is another effective divisor on X such that D and E intersect at x and have no common components through x, then the intersection multiplicity of D and E at x is given by

$$(D.E)_x = \sum_{p \in X^*} m_p(D) m_p(E)[k(p) : k],$$

where the sum is over the infinitely near points p of x in common to D and E, and $k(p)$ is the residue field at the point p. If k is algebraically closed, then each infinitely near point p is rational over k, so the integer $[k(p) : k]$ is equal to 1.

If D is a reduced curve on X with $x \in D$, then the 'double point number' $\delta_x(D)$ is given by

$$\delta_x(D) = \tfrac{1}{2} \sum_{p \in X^*} m_p(D)(m_p(D) - 1)[k(p) : k], \tag{1}$$

and we have the *proximity relations*

$$m_p(D) = \sum_{p'} m_{p'}(D)[k(p') : k(p)], \tag{2}$$

where the sum is over all the infinitely near points p' proximate to p.

LEMMA 1.1. *Let C be an irreducible curve germ over an algebraically closed field; suppose that repeatedly blowing up C produces the sequence $\{p_i \mid i \geqslant 0\}$ of infinitely near points, and that p_{n+2} is not proximate to any p_i with $i < n$. Then there are numbers a and b, determined solely by the proximity relations, such that for any curve Γ through p_{n+1},*

$$m_0(\Gamma) = am_n(\Gamma) + bm_{n+1}(\Gamma).$$

Proof. Suppose inductively that we have found a_r and b_r such that for all Γ through p_{n+1} we have $m_{n-r}(\Gamma) = a_r m_n(\Gamma) + b_r m_{n+1}(\Gamma)$; the induction starts trivially with $a_{-1} = 0$, $b_{-1} = 1$, $a_0 = 1$, $b_0 = 0$. Then the multiplicity at p_{n-r-1} is the sum of the multiplicities at all points proximate to that one, which must be $p_{n-r}, \ldots, p_{n-s}$ where $s \geqslant -1$ since it follows from our hypothesis that no p_q with $q > n+1$ can be proximate to any p_i with $i < n$. Thus we may take $a_{r+1} = \sum_s^r a_i$ and similarly for b_{r+1}. □

1.2. *Resolving base points of pencils of curves*

We now collect some well-known results on resolutions of pencils on a smooth projective surface X over an algebraically closed field k. Let Λ be a pencil of curves $\{\Gamma_t\}$ on X which has no fixed component, and let $x \in X$ be any point of intersection of two, and hence of all, curves in the pencil: such points are called *base points*. Write $m_x(\Lambda) = \min\{m_x(\Gamma_t) \mid t \in P^1\}$. Then $m_x(\Lambda) = m_x(\Gamma_t)$ for all but finitely many $t \in P^1$.

Let $\pi \colon \widetilde{X} \to X$ be the blow-up of X at x, and E the exceptional divisor of π. Then the *total transform* $\pi^*(\Lambda) = \{\pi^*(\Gamma_t) \mid \Gamma_t \in \Lambda\}$ of Λ is a linear system with $m_x(\Lambda)E$ as fixed component. Removing this gives the *strict transform* $\Lambda_1 = \{\pi^*(\Gamma_t) - m_x(\Lambda)E \mid t \in P^1\}$ of Λ, which has no fixed component.

Let $q \in X_r$ be an infinitely near point of p obtained by a finite sequence of blowings-up $\pi_i \colon X_{i+1} \to X_i$ (with $X_0 = X$). Set $m_q(\Lambda) := m_q(\Lambda_r)$ where Λ_r is the strict transform of Λ on X_r. We say that q is a base point of Λ if $m_q(\Lambda) > 0$.

Inductively blow up at a base point of the pencil, take the strict transform of the pencil, and continue. Since for two members of the pencil with multiplicity $m_x(\Lambda)$ at x, blowing up x reduces their intersection multiplicity there by $m_x(\Lambda)^2$, the blow-up reduces the total intersection number of these two, and hence of any two, members of the pencil by $m_x(\Lambda)^2$. As the original intersection number is finite, we may continue till no base points remain. Write B for the set of base points (including infinitely near ones) of Λ: then this is a finite set.

At the end, we have a smooth projective surface Y with a well-defined map $\pi \colon Y \to P^1$, whose fibres $Y_t = \pi^{-1}(t)$ project to the curves Γ_t of the original pencil.

In local coordinates (u, v) on Y we may write $\pi(u, v) = (f(u, v) : g(u, v))$ where f and g are algebraic functions on Y, smooth at the point in question; then Y_t is given by $f = tg$.

An exceptional curve in Y may project to a single point of P^1 and so lie in a fibre, when we call it *vertical*, or may map onto P^1, in which case we call it *horizontal* or *dicritical*.

We now discuss the creation of a single exceptional curve E_α: write $m = m_\alpha := m_{e_\alpha}(\Lambda)$. Take local coordinates in X_α with e_α at $(0, 0)$. Then we can write $f = \sum_m^\infty f_k$ as a sum of homogeneous components, and similarly for g. The m-jets f_m and g_m define a pencil of binary m-ics. The curve E_α has parameters $(\xi : \eta) = (u : v)$ whose projection to P^1 is given by $f_m = tg_m$. Thus E_α is vertical if and only if some member of the pencil of binary forms vanishes identically.

To describe the geometry when E_α is horizontal, first note that any common factor of f_m and g_m produces a base point of the blown-up pencil on E_α. To determine the multiplicities of these requires an examination of terms of higher order in f and g: for example, $x = 0$ defines a singular point of the strict transform in Y_α of $f = 0$ if and only if y^2 divides f_m and y divides f_{m+1}.

If the degree of the highest common factor of f_m and g_m is h then the degree of the projection $\pi_\alpha: E_\alpha \to P^1$ is $m - h$. If f_m/g_m is a function of $(u/v)^p$, where p is the characteristic, then all values of t give multiple roots, and to study the geometry it is necessary to factorise $\pi_\alpha = \pi_i \circ \pi_s$ where $\pi_i(t) = t^{p^r}$ for some r, and π_s is not a function of $(u/v)^p$: here $q = p^r$ is called the *degree of inseparability*. Equivalently, since k is algebraically closed, we may write $f_m/g_m = \phi^q$ and factorise $\pi_\alpha = \pi'_\alpha \circ F_q$, where $F_q: E_\alpha \to D$ is purely inseparable of degree q and $\pi'_\alpha: D \to P^1$ is separable, of degree $q^{-1}(m - h)$. We will refer to the points of P^1 over which π'_α is ramified as the *branch values* of E_α.

If $q > 1$, all points of E_α are singular for π_α, and the branching behaviour is best studied via π'_α. The number of branch points of π'_α (counting multiplicities appropriately) is $2q^{-1}(m - h) - 2$. The intersections of E_α with the fibre Y_t are the preimages of $t \in P^1$ under π_α: for almost all t this gives $q^{-1}(m - h)$ distinct points, but for t a branch value of π'_α the number is decreased.

Now suppose E_α is vertical: we may suppose $g_m \equiv 0$. Let the least order terms in g have degree $m + k$. Then E_α lies in the fibre Y_s over the point s where $t = \infty$, and has multiplicity k as component of that fibre. The base points of the pencil in the surface Y_α are the points on E_α where f_m vanishes, and E_α intersects its complement Z in the corresponding fibre $Y'_{s,\alpha}$ of the map $\pi_\alpha: Y_\alpha \to P^1$ in the points where $g_{m+k} = 0$. Although the roots of g_{m+k} need not be distinct, the intersection number of Z and E_α is $m + k$.

LEMMA 1.2. *Any exceptional curve of the first kind contained in a fibre Y_t of Y must be a component of $\widetilde{\Gamma}_t$.*

For at the creation of a vertical exceptional curve E_α there are at least some branch points on it, so in Y each such curve has been blown up at least once more, and so is not exceptional of the first kind.

1.3. *Cohomology and Euler characteristics of curves*

An Euler characteristic is an alternating sum of Betti numbers. In finite characteristic, it is necessary to use étale cohomology groups, which were introduced by Grothendieck [14]. We recall that one first chooses a prime ℓ different from p, constructs cohomology groups $H^r_{\text{ét}}(X; \mathbb{Z}/\ell^k)$ with finite coefficient groups, and then takes the inverse limit $H^r_{\text{ét}}(X; \widehat{\mathbb{Z}}_\ell)$. For our purposes we need only consider the case when X is a proper scheme.

Let C be a reduced, complete algebraic curve over an algebraically closed field k. Let $C = \bigcup_{i=1}^h C_i$ be its decomposition into irreducible components; write also s for the number of connected components of C. Let $\overline{C}_i$ be the normalisation of C_i, $\overline{C}$ be the disjoint union of the $\overline{C}_i$ and n: $\overline{C} \to C$ be the canonical projection. Write g_i for the genus of $\overline{C}_i$ and $g := \sum g_i$.

For each closed point x of C set $\delta_x(C) := \dim_k n_*(\mathcal{O}_{\overline{C}})_x/\mathcal{O}_{C,x}$ (see also (1)) and write $\delta(C) := \sum_{x\in|C|} \delta_x(C)$ where $x \in |C|$ means that x runs through closed points of C. These numbers are finite because n is a finite map and $n_*(\mathcal{O}_{\overline{C}})/\mathcal{O}_C$ is a coherent sheaf supported on a finite set of points. If $r_x(C)$ denotes the number of analytic branches of C at x, let $\delta'_x(C) := r_x(C) - 1$ and $\delta'(C) := \sum_{x\in|C|} \delta'_x(C)$. We use $\mu_x(C) := 2\delta_x(C) - \delta'_x(C)$ as a definition of the Milnor number of C at x and write $\mu(C) := \sum_{x\in|C|} \mu_x(C) = 2\delta(C) - \delta'(C)$ for the total Milnor number of C. Note that, in distinction to the characteristic zero case, this number cannot easily be calculated from a local equation for C: we discuss this point fully below.

Other invariants of C may now be expressed in terms of the above. Dolgachev [11] showed that the étale cohomology groups of C are free of rank $\beta_i(C)$, where the Betti numbers $\beta_i(C)$ are given by

$$\beta_0(C) = s, \quad \beta_1(C) = 2g + \delta'(C) - h + s, \quad \beta_2(C) = h; \quad \beta_i(C) = 0 \text{ for } i > 0.$$

Thus the topological Euler–Poincaré characteristic (l-adic Euler–Poincaré characteristic) of C is

$$\chi(C) = \chi_{\text{ét}}(C) = \sum (-1)^i \beta_i(C) = 2h - 2g - \delta'(C).$$

Thus if p: $C' \to C$ is a birational map (for example, obtained via blowing up), then the values of h and g agree for C and C', so $\chi(C') - \chi(C) = \delta'(C) - \delta'(C')$. This can be written in the more familiar form $\sum_{x\in C} \#(p^{-1}(x) - 1)$.

The cohomology of the sheaf $\mathcal{O}_C$ is given in terms of the above invariants (via the Leray spectral sequence for the morphism n) by

$$\dim_k H^0(C, \mathcal{O}_C) = s, \qquad p_a(C) := \dim_k H^1(C, \mathcal{O}_C) = g + \delta(C) - h + s,$$

so $\chi(\mathcal{O}_C) = h - g - \delta(C)$, and we obtain the useful formula

$$\chi(C) = 2\chi(\mathcal{O}_C) + \mu(C). \tag{3}$$

We will find the invariant $\chi(\mathcal{O}_C)$ convenient since [17, p. 261] it is constant on flat families. Thus for curves in a smooth surface Y, $\chi(\mathcal{O}_C)$ depends only on the equivalence class of the divisor of C, and is easy to compute. In particular – a fact of which we will make constant use – it takes the same value for any two fibres of a map $Y \to S$. We also have the adjunction formula

$$0 = 2\chi(\mathcal{O}_C) + C^2 + C.K_Y, \tag{4}$$

where . denotes intersection of divisors and K_Y the canonical divisor; and the formula

$$\chi(\mathcal{O}_{A+B}) = \chi(\mathcal{O}_A) + \chi(\mathcal{O}_B) - A.B, \tag{5}$$

which follows from the adjunction formula.

2. *Fibrations with smooth general fibre*

We now suppose that $\pi: Y \to S$ is a map with smooth general fibre from a smooth surface Y to a smooth curve S, all defined over the algebraically closed field k. Over each closed point $s \in S$ we have the fibre Y_s; over the generic point we also have a fibre, which is a variety defined over $k(S)$; we denote it by Y_{gen}. It is usually more geometrical to work over algebraically closed fields; the variety obtained by extending the ground field to the separable algebraic closure is called the *geometric generic fibre*: we will denote it by $Y_{\text{ggen}} := Y_{\text{gen}} \otimes_{k(S)} \overline{k(S)}$. Our present hypothesis is that Y_s is smooth for all but finitely many s, or equivalently that Y_{ggen} is smooth.

In characteristic zero, the additive property of the Euler characteristic provides a formula

$$\chi(Y) - \chi(Y_{\text{ggen}})\chi(S) = \sum_s (\chi(Y_s) - \chi(Y_{\text{ggen}})),$$

where there are only finitely many non-vanishing terms on the right-hand side. For reduced fibres we may write (using (3)) $\chi(Y_s) - \chi(Y_{\text{ggen}}) = \mu(Y_s)$; for non-reduced fibres various formulae may be inferred by considering the underlying reduced curve $Y_{s,\text{red}}$ and using the facts that $\chi(Y_s) = \chi(Y_{s,\text{red}}) = 2\chi(\mathcal{O}_{Y_{s,\text{red}}}) + \mu(Y_{s,\text{red}})$. In this section we discuss what becomes of these formulae in the characteristic p case. This involves deep results of Grothendieck and others: it seems worth presenting an account here since many results simplify considerably in this relatively simple situation.

2.1. *Algebra*

Fix a rational point $s \in S$, and write $\mathcal{O}_{S,s}$ for the local ring of S at s, with field $k(S)$ of quotients with separable (algebraic) closure $\overline{k(S)}$; write $\widehat{G}$ for the Galois group of $\overline{k(S)}$ over $k(S)$. Choose an extension $\overline{v}_s$ to $\overline{k(S)}$ of the discrete valuation v_s of $k(S)$ with valuation ring $\mathcal{O}_{S,s}$, and write G_s, or G for short, for the inertia group of $\overline{v}_s$: then G_s is a pro-p-group.

The group $\widehat{G}$ acts on the geometric generic curve Y_{ggen} and hence on its (étale) cohomology group $H := H^1_{\text{ét}}(Y_{\text{ggen}}; \widehat{\mathbb{Z}}_\ell)$, where ℓ is a prime different from p. As a module over $\widehat{\mathbb{Z}}_\ell$, H is torsion-free with rank $2g$, where g is the genus of Y_{ggen}. The kernel of the projection $\mathrm{GL}_{2g}(\widehat{\mathbb{Z}}_\ell) \to \mathrm{GL}_{2g}(\mathbb{Z}/\ell\mathbb{Z})$ is a pro-ℓ-group, and it follows that the image of G_s in the former group maps isomorphically to the second, so in particular is finite. Thus the representation of G_s on $H^1_{\text{ét}}(Y_{\text{ggen}}; \mathbb{Z}/\ell\mathbb{Z})$ determines that on H up to isomorphism.

Let M be any finite G-module annihilated by ℓ. Then a subgroup of finite index in G acts trivially on M: write G_0 for the quotient of G which acts effectively, and k_M for the finite extension of $k(S)$ corresponding to G_0 by Galois duality. Write v_M for the restriction to k_M of the valuation $\overline{v}_s$ of $\overline{k(S)}$: normalise so that v_M has value group $\mathbb{Z}$. Choose a prime element ϖ of k_M, so $v_M(\varpi) = 1$. We have the ramification groups

$$G_i := \{\sigma \in G_0 \mid v_M(\sigma(\varpi) - \varpi) \geqslant i + 1\}.$$

The *Artin character* of G_0 is now defined by

$$\alpha(\sigma) = -v_M(\sigma(\varpi) - \varpi) \text{ if } \sigma \neq 1; \qquad \sum_{\sigma \in G_0} \alpha(\sigma) = 0.$$

The *Swan character* β is obtained by subtracting from α the character of the representation of G_0 on the augmentation ideal in $\mathbb{C}[G_0]$. According to Serre (see, for example, [**26**]), there is a projective $\widehat{\mathbb{Z}_\ell}[G_0]$-module P such that β is the character of the representation on $P \otimes \widehat{\mathbb{Q}_\ell}$. Then 'Serre's measure of the wildness' of M (also called the 'Swan conductor') is

$$\mathrm{Sw}(M, G) := \dim_{\mathbb{Z}/\ell\mathbb{Z}} \mathrm{Hom}_{\widehat{\mathbb{Z}_\ell}[G_0]}(P, M).$$

Below we will write $\mathrm{Sw}(M, s)$ for $\mathrm{Sw}(M, G_s)$. It is shown in [**22**] that, if M^i denotes the fixed set of G_i acting on M, then

$$\mathrm{Sw}(M, G) = \sum_{i=1}^{\infty} \frac{|G_i|}{|G_0|} \dim_{\mathbb{Z}/\ell\mathbb{Z}}\left(\frac{M}{M^i}\right).$$

2.2. *The Ogg–Šafarevič–Grothendieck formula*

The main ingredient of the formulae we are about to describe was first obtained by Ogg [**21**], and developed in full by Grothendieck [**15**]; see also Šafarevič [**24**] and the account [**23**] of Raynaud.

Let S be (as above) a smooth curve, and F a sheaf over S of abelian groups annihilated by ℓ: write F_{ggen} for the stalk over the geometric generic point of S. Then the formula is [**15**, 7.2]

$$\chi(S; F) - \chi(S) \dim F_{\mathrm{ggen}} = \sum_{s \in |S|} (\dim F_s - \dim F_{\mathrm{ggen}} - \mathrm{Sw}(F_{\mathrm{ggen}}, s)), \qquad (6)$$

where all dimensions are over $\mathbb{Z}/\ell$. Strictly, we should take F as an element of the derived category of such sheaves; the dimensions must then be replaced by Euler characteristics.

The Leray spectral sequence of $\pi\colon Y \to S$ for the constant sheaf $\mathbb{Z}/\ell$ gives the calculation

$$\begin{aligned}\chi(Y; \mathbb{Z}/\ell) &= \sum_{p,q} (-1)^{p+q} \dim H^p(S; R^q\pi_*(\mathbb{Z}/\ell)) \\ &= \sum_{q} (-1)^q \chi(S; R^q\pi_*(\mathbb{Z}/\ell)). \qquad (7)\end{aligned}$$

Applying (6) yields

$$\chi(S; R^q\pi_*(\mathbb{Z}/\ell)) - \chi(S) \dim(R^q\pi_*(\mathbb{Z}/\ell))_{\mathrm{ggen}} = \sum_{s \in |S|} A_s^q,$$

where A_s^q is given by

$$\begin{aligned}A_s^q &:= \dim(R^q\pi_*(\mathbb{Z}/\ell))_s - \dim(R^q\pi_*(\mathbb{Z}/\ell))_{\mathrm{ggen}} - \mathrm{Sw}((R^q\pi_*(\mathbb{Z}/\ell))_{\mathrm{ggen}}, s) \\ &= \dim H^q(Y_s; \mathbb{Z}/\ell) - \dim H^q(Y_{\mathrm{ggen}}; \mathbb{Z}/\ell) - \mathrm{Sw}(H^q(Y_{\mathrm{ggen}}; \mathbb{Z}/\ell), s).\end{aligned}$$

From now on, we omit explicit mention of the coefficients, which will remain $\mathbb{Z}/\ell$ throughout. Substituting in (7) gives

$$\chi(Y) = \chi(S)\chi(Y_{\mathrm{ggen}}) + \sum_{s \in |S|} (\chi(Y_s) - \chi(Y_{\mathrm{ggen}}) - \mathrm{Sw}(H^*(Y_{\mathrm{ggen}}), s)),$$

where $\mathrm{Sw}(H^*(Y_{\mathrm{ggen}}), s) := \sum_q (-1)^q \mathrm{Sw}(H^q(Y_{\mathrm{ggen}}), s)$. This is essentially the main result of [**12**].

When the fibres are curves, this simplifies since, by [12, 1.6], if the fibres of π are smooth of dimension n, then $\mathrm{Sw}(H^q(Y_{\mathrm{ggen}}), s)$ vanishes unless $0 < q < 2n$, so as $n = 1$, the only non-vanishing term is for $q = 1$. Thus we have

$$\chi(Y) = \chi(S)\chi(Y_{\mathrm{ggen}}) + \sum_{s \in |S|} (\chi(Y_s) - \chi(Y_{\mathrm{ggen}}) + \mathrm{Sw}(H^1(Y_{\mathrm{ggen}}), s)), \tag{8}$$

2.3. *Vanishing cycles*

A general theory of vanishing cycles, due to Grothendieck, is developed by Deligne in [9]. Again we require only a small part of the general theory. Fix a (rational) point $s \in |S|$ (this already simplifies half the notation); the theory is set in the context of the spectrum of the henselisation of $\mathcal{O}_{S,s}$, which we denote by S_{loc}: it contains the closed point s, the generic point η, and the geometric generic point $\bar{\eta}$; and we have a map $S_{\mathrm{loc}} \to S$, inducing $Y_{\mathrm{loc}} \to Y$, say, with fibres Y_s, Y_η and $Y_{\bar{\eta}}$ which we can identify with the Y_s, Y_{gen} and Y_{ggen} discussed previously.

The first step is to define a functor Ψ from (complexes of) sheaves over Y_{loc}, annihilated by some ℓ^n, to (complexes of) sheaves over $Y_s \times S_{\mathrm{loc}}$. To achieve this, one must observe that giving a sheaf K over $Y_s \times S_{\mathrm{loc}}$ is equivalent to giving a triple (K_s, K_η, ϕ) with K_s and K_η sheaves over Y_s and ϕ a morphism from K_s to K_η with image contained in the invariants under the action on the latter of the group G_s or Galois group G_0. There is then a (split) exact sequence $0 \to (s \circ p)^* K_s \to K_\eta \to \Phi(K) \to 0$, inducing an exact sequence [9, 1.4.2.2] of cohomology of Y_s.

Given a sheaf F, Deligne defines (1.3 loc. cit.) $\Psi F_s := F|_s$; ΨF_η is the pullback of the push forward to Y of $F|_{\bar{\eta}}$, and ϕ is constructed using adjunction morphisms. The above short exact sequence of sheaves must be considered in the derived category as an exact triangle (2.1.2.4 loc. cit.). Further work leads to the identification (2.1.8.9 loc. cit.) of its cohomology exact sequence with

$$\ldots \to H^i(Y_s; F) \to H^i(Y_{\bar{\eta}}; F) \to H^i(Y_s; \Phi) \to H^{i+1}(Y_s; F) \to \ldots,$$

where Φ denotes $\Phi(\Psi F)$. It is also shown that Φ is supported on the singular set of the map f, and (2.4.2 loc. cit.) that if F is a constructible sheaf of $\mathbb{Z}/\ell^n$-modules (with $\ell \neq p$) which is locally constant on a neighbourhood of x in Y_s, then the stalk Φ_x of Φ at x is a finite abelian group.

We are interested in the case when F is a constant sheaf $\mathbb{Z}/\ell^n$ and the fibres Y_s are curves. The exact sequence shows that $\chi(Y_{\bar{\eta}}) - \chi(Y_s) = \chi(Y_s; \Phi)$. Since $\pi_*(\mathbb{Z}/\ell) \cong (\mathbb{Z}/\ell)$, $H^0(Y_s; \Phi)$ vanishes. Also $H^2(Y_s; \Phi)$ is trivial: we may argue by duality, or, perhaps more convincingly, by noting that at least one component of Y_s must have multiplicity prime to ℓ. So there is only one non-vanishing module of vanishing cycles, and we write V_s for $H^1(Y_s; \Phi)$. Thus

$$\chi(Y_{\bar{\eta}}) - \chi(Y_s) = -\dim V_s. \tag{9}$$

Since forming Swan conductors is purely algebraic and additive, and gives zero on the cohomology of Y_s where the action of G is trivial, we have

$$\mathrm{Sw}(H^*(Y_{\mathrm{ggen}}), s) = \mathrm{Sw}(H^*(Y_s; \Phi), s) = -\mathrm{Sw}(H^1(Y_s; \Phi), s) = -\mathrm{Sw}(V_s, s). \tag{10}$$

Substituting in (8), we obtain

$$\chi(Y) = \chi(S)\chi(Y_{\mathrm{ggen}}) + \sum_{s \in |S|} (\dim V_s + \mathrm{Sw}(V_s, s)). \tag{11}$$

Since Φ is supported on the singular set, if Y_s is reduced, and so has isolated singularities, Φ is supported on a finite set; so V_s splits as a sum of local contributions $V_s = \bigoplus V_x$.

2.4. *The degree of the discriminant*

We turn to global consideration of the map $\pi: Y \to S$ with Y and S smooth and complete and with the generic fibre also smooth, and apply Example 14.1.5 of Fulton's book [**13**]. We have a proper morphism π; the induced map $d\pi: T_Y \to \pi^* T_S$ determines a section σ of $\pi^* T_S \otimes T_Y^\vee$. The top Chern class $Z(\sigma)$ is a 0-cycle class on the singular set of π, with degree $\deg Z(\sigma) = \chi(Y) - \chi(S)\chi(Y_{\text{ggen}})$. The image under π of the class of $Z(\sigma)$ is a divisor D_π on S; thus

$$\deg D_\pi = \deg Z(\sigma) = \chi(Y) - \chi(S)\chi(Y_{\text{ggen}}).$$

It was shown by Bloch [**2**], sharpening a result of Deligne [**10**] and also giving an interpretation of (8), that the degree of D_π at a point $s \in |S|$ is

$$D(s) := \deg_s(D_\pi) = \chi(Y_s) - \chi(Y_{\text{ggen}}) - \text{Sw}(H^*(Y_{\text{ggen}}, s)). \tag{12}$$

Bloch also interpreted this number as a cycle-theoretic self-intersection of the diagonal Δ_Y.

It follows from (9) and (10) that in the case of curves, this in turn reduces to

$$D(s) = \dim V_s + \text{Sw}(V_s, s). \tag{13}$$

Thus $D(s) \geqslant 0$, and $D(s)$ vanishes if and only if V_s is trivial.

We first discuss the case of an isolated critical point. In [**10**], Deligne defines the numbers of ordinary, wild or total vanishing cycles at an isolated singular point x to be respectively the numbers $\dim V_x$, $\dim \text{Sw}(V_x, s)$ and their sum, and proves (Theorem 2.4 loc. cit.) that the total number of vanishing cycles at x is equal to the Milnor number of π at x, defined in the usual way by taking local coordinates (u, v) on Y at x and t on S at $s = f(x)$ and setting $f := t \circ \pi$ and

$$\mu_x(\pi) := \dim \mathcal{O}_{u,v} / \langle \partial f/\partial u, \partial f/\partial v \rangle.$$

That this is also the local degree of $Z(\sigma)$ is shown by Bloch.

The relation between the two definitions of Milnor number at an isolated singularity is now clarified as follows. The Milnor number $\mu_x(Y_s)$ of the fibre at x is equal to $\chi(Y_s) - \chi(Y_{\text{ggen}})$, and hence to $\dim V_s$: the 'ordinary number of vanishing cycles'. The inertia group G_s of $s = \pi(x)$ acts on this space, and the Milnor number of the map π is given by

$$\mu_x(\pi) = \mu_x(Y_s) + \text{Sw}(V_x, s). \tag{14}$$

If the fibre Y_s is reduced, $D(s)$ is the sum of the $\mu_x(\pi)$ over singular points $x \in |Y_s|$: the decomposition into tame and wild cycles is given by (14).

Suppose in addition that $D(s) = 0$. Then there are no singular points; if Y_{ggen} is connected, so is Y_s, and it has the same genus. Over $\mathbb{C}$, we can then apply Ehresmann's fibration theorem to deduce that π is locally topologically trivial at s. In finite characteristic, π is still locally trivial in some sense which we do not attempt to make precise.

If Y_s is not reduced, to obtain a local formula we need a term corresponding to the 1-dimensional part of the singular locus. Globally, Iversen [**18**] obtained a formula which may be written as

$$\chi(Y) - \chi(Y_{\text{ggen}})\chi(S) = K_Y . R - R^2 + \sum_{x \in |Y|} \mu_x^{\text{red}}(\pi), \tag{15}$$

where R denotes the ramification divisor, K_Y the canonical class, and $\mu_x^{\text{red}}(\pi)$ is defined as follows. As in the case when x is an isolated singular point, we take local coordinates (u, v) on Y at x and t on S at $s = \pi(x)$, and set $f = t \circ \pi$. The partial derivatives $\partial f / \partial u$, $\partial f / \partial v$ do not generate an ideal of finite codimension in the local ring $\mathcal{O}_x$: their highest common factor h vanishes along the singular locus. We set $a^u := (\partial f / \partial u)/h$, $a^v := (\partial f / \partial v)/h$, and define

$$\mu_x^{\text{red}}(\pi) := \dim h\mathcal{O}_{u,v} / \langle \partial f / \partial u, \partial f / \partial v \rangle = \dim \mathcal{O}_{u,v} / \langle a^u, a^v \rangle.$$

The right-hand side of (15) is easily expressed as a sum over fibres, and the natural interpretation was achieved by Sun **[28]** by showing that

$$D(s) = K_Y . R_s - R_s^2 + \sum_{x \in |Y_s|} \mu_x^{\text{red}}(\pi), \tag{16}$$

where R_s denotes the part of R lying in the fibre over s.

The divisor R is supported on the union of multiple components of the fibres. Let C_i be a component of Y_s, with multiplicity $m_i > 1$. Let r_i be the multiplicity of C_i as a component of R. In local coordinates at a point of T we may write

$$(t \circ \pi)(u, v) = f(u, v) = u^{m_i} \phi(u, v).$$

The definition of R amounts to taking it as defined by the highest common factor of $\partial f / \partial u$ and $\partial f / \partial v$.

LEMMA 2.1. *If m_i is not divisible by p (for example, if $p = 0$) then $r_i = m_i - 1$. If $p \mid m_i$ then $r_i = m_i$* unless *we can express ϕ in the form $\alpha(v^p) + u^2 \beta(u, v)$, when $r_i > m_i$.*

This follows by an easy calculation. In the first case we have non-zero terms involving $u^{m_i - 1}$; in the second, if the coefficients of all terms $u^{m_i} v^*$ in both $\partial f / \partial u$ and $\partial f / \partial v$ vanish, ϕ must be as stated.

In the case when C_i is a vertical exceptional curve arising in a blowing-up, we take local coordinates and suppose, reverting to the notation of 1.2 and setting $\pi(x) = (f(x) : g(x))$, that f and g have respective orders $m + M$ and m. Substituting $v = uw$ gives $f/g = u^M (f_{m+M}(1, w) + O(u))/(g_m(1, w) + O(u))$, where $O(u)$ denotes terms divisible by u. Thus the above invariant m_i is here identified with M. For the final case in the lemma to arise we require that

$$\frac{f_{m+M}(1, w)}{g_m(1, w)} + O(u) = \alpha(w^p) + u^2 \beta,$$

and hence that for some $\gamma(w)$ we have $f_{m+M}(1, w) \cong \gamma^p g_m(1, w)$.

It would be feasible to push these calculations further.

2.5. *The Euler characteristic of the fibre*

We now study a particular (non-reduced) fibre Y_s. Then we cannot regard the fibration as locally trivial at s, and so we expect that $D(s) > 0$. By (13), this condition is equivalent to $\chi(Y_{\text{ggen}}) < \chi(Y_s) = \chi(Y_{s,\text{red}})$. We next investigate whether indeed these inequalities hold.

We begin with some notation. Let Y_s have components C_i, and $Y_s = \sum m_i C_i$ as divisors. Then the underlying reduced curve is $Y_{s,\text{red}} = \sum C_i$. Also write $R_s = \sum r_i C_i$. Thus in the characteristic 0 case, $Y_s = R_s + Y_{s,\text{red}}$; in general, $R_s + Y_{s,\text{red}} - Y_s$ has non-negative coefficients $r_i + 1 - m_i$ which equal 0 only when m_i is not divisible by p.

Since Y_s is numerically equivalent to Y_{ggen} which is disjoint from Y_s, we have, for each j, $0 = Y_s . C_j = \sum m_i C_i . C_j$ and in particular, $Y_s^2 = 0$. Below we shall also use without further mention the invariance $\chi(\mathcal{O}_{Y_s}) = \chi(\mathcal{O}_{Y_{\text{ggen}}})$, the formula (3) and the adjunction formula (4).

Proposition 2.2. *Assume that Y_{ggen} is connected, and that Y_s contains no exceptional curve of the first kind. Then for any divisor A on Y_s with $0 \leqslant A < Y_s$, either $A = 0$ and Y_{ggen} and Y_s are smooth and rational; or $\chi(\mathcal{O}_A) \geqslant \chi(\mathcal{O}_{Y_s})$, with equality only if $A = \lambda Y_s$ with $0 \leqslant \lambda < 1$ and Y_{ggen} is elliptic.*

Proof. We first briefly recall the (standard) proof that the intersection form restricted to the fibre Y_s is negative semi-definite, with radical generated rationally by Y_s. We have $C_i . C_j \geqslant 0$ for $i \neq j$, and $0 = C_i . Y_{\text{ggen}} = C_i . Y_s = \sum_j m_j C_i . C_j$. Now for any coefficients $x_i \in \mathbb{Q}$,

$$\left(\sum_i x_i C_i\right)^2 = \sum_i x_i^2 C_i^2 + \sum_{i\neq j} x_i x_j C_i . C_j$$

$$= \sum_i \frac{x_i^2}{m_i} \sum_{j\neq i} (-m_j C_i . C_j) + \sum_{i\neq j} x_i x_j C_i . C_j,$$

which reduces to

$$\sum_{i<j} -\frac{(m_j x_i - m_i x_j)^2}{2 m_i m_j} C_i . C_j \leqslant 0.$$

For equality to hold, $m_j x_i - m_i x_j$ must vanish whenever $C_i . C_j > 0$, that is, x_i/m_i takes equal values for components C_i and C_j that intersect. Since the fibre is connected (Principle of Connectedness, [17]), all values are equal; so for some λ we have $x_i = \lambda m_i$ for each i.

Next let C be a reduced, irreducible component of Y_s, and consider $C . K_Y$.

We have $\chi(\mathcal{O}_C) = 1$ if C is smooth and rational, $\chi(\mathcal{O}_C) = 0$ if C is either nodal or cuspidal rational or smooth elliptic, and $\chi(\mathcal{O}_C) < 0$ otherwise. If C is a component of Y_s, then since $C . K_Y = -C^2 - 2\chi(\mathcal{O}_C)$ and $C^2 \leqslant 0$, we deduce that if $C . K_Y < 0$ then either

(i) $\chi(\mathcal{O}_C) = 1$ and $C^2 = -1$, so that C is smooth and rational and is an exceptional curve of the first kind contained in Y_s: but by hypothesis this does not exist; or

(ii) $\chi(\mathcal{O}_C) = 1$ and $C^2 = 0$, so that C is the whole fibre. If $Y_s = mC$, we obtain $\chi(\mathcal{O}_{Y_{\text{ggen}}}) = \chi(\mathcal{O}_{Y_s}) = m$, so that m must be 1, the fibration rational and Y_s a smooth fibre. Since $0 \leqslant A < Y_s$ we now have $A = 0$, and the first alternative holds.

Thus from now on we may suppose that $C . K_Y \geqslant 0$. Moreover, if $C . K_Y = 0$ then either

(i) $\chi(\mathcal{O}_C) = 1$ and $C^2 = -2$, so C is smooth and rational: C is a '(−2)-curve', or

(ii) $\chi(\mathcal{O}_C) = 0$ and $C^2 = 0$; hence Y_s is a multiple of C, and the entire fibre reduces to the curve C.

Now let $A = \sum a_i C_i$ with $0 \leqslant a_i \leqslant m_i$. By the adjunction formula,

$$2(\chi(\mathcal{O}_A) - \chi(\mathcal{O}_{Y_s})) = Y_s \,.\, Y_s + Y_s \,.\, K_Y - A \,.\, A - A \,.\, K_Y.$$

But the intersection form restricted to a fibre is negative semi-definite, so $A \,.\, A \leqslant 0$, while $Y_s \,.\, Y_s = 0$. We thus obtain

$$2(\chi(\mathcal{O}_A) - \chi(\mathcal{O}_{Y_s})) = (-A \,.\, A) + \sum_i (m_i - a_i) C_i \,.\, K_Y.$$

Here each term on the right is non-negative. Thus for the sum to be 0, each term must vanish. Since $A^2 = 0$, A is a (rational) multiple of Y_s, so for some λ, $a_i = \lambda m_i$ for each i. It follows from our hypothesis that $\lambda < 1$, so $m_i > a_i$ for each i. Hence each $C_i \,.\, K_Y$ must vanish, and we have $Y_s \,.\, K_Y = 0$; thus the fibration is elliptic. □

We can be more precise about the case of equality.

COROLLARY 2.3. *If equality holds, Y_s has Kodaira type mI_b for some $m \geqslant 2$ and $b \geqslant 0$. Thus $Y_s = m\Theta_b$ for some b, where*

Θ_0 is smooth elliptic,

Θ_1 is nodal rational,

Θ_2 consists of two smooth rational curves meeting transversely in two points,

for $b \geqslant 3$, Θ_b is a cycle of b (−2)-curves, each component meeting the next.

This follows from the enumeration of types of (non-reduced) fibre in elliptic fibrations by results of Kodaira **[19]** in characteristic 0 and Bombieri and Mumford **[3]** in general.

We next deal with exceptional curves contained in the fibre.

PROPOSITION 2.4. *Assume that Y_{ggen} is connected. Suppose Z is obtained by successively collapsing exceptional curves of the first kind in the fibre over s. Then $(\chi(\mathcal{O}_{Y_{s,\text{red}}}) - \chi(\mathcal{O}_{Y_s})) \geqslant (\chi(\mathcal{O}_{Z_{s,\text{red}}}) - \chi(\mathcal{O}_{Z_s}))$, and hence is non-negative.*

Proof. Suppose there is an exceptional curve of the first kind $E \subset Y_s$. Write $Y_{s,\text{red}} = B + E$, and let $M := E \,.\, B$. Collapsing E to a point gives a surface Y', a factorisation $Y \to Y' \to S$ of π, and hence a fibre Y'_s with $Y'_{s,\text{red}} = B'$, the image of B, so that B' has multiplicity M at the image e of E and hence strict transform $B + ME$. The standard rules for blowings-up give

$$B \,.\, K_Y = B' \,.\, K_{Y'} + M, \qquad B^2 = B'^2 - M^2;$$

so

$$\chi(\mathcal{O}_B) = \chi(\mathcal{O}_{B'}) + \tfrac{1}{2}(M^2 - M).$$

Hence

$$\chi(\mathcal{O}_{Y_{s,\text{red}}}) = \chi(\mathcal{O}_B) + \chi(\mathcal{O}_E) - B \,.\, E = \chi(\mathcal{O}_{B'}) + \tfrac{1}{2}(M^2 - M) + 1 - M,$$

which is equal to $\chi(\mathcal{O}_{Y'_{s,\text{red}}}) + \frac{1}{2}(M-1)(M-2) \geqslant \chi(\mathcal{O}_{Y'_{s,\text{red}}})$. If Y'_s also contains an exceptional curve, we may repeat the process, which must terminate at some surface Z, say, with fibre Z_s. It follows by induction that $\chi(\mathcal{O}_{Y_{s,\mathrm{red}}}) \geqslant \chi(\mathcal{O}_{Z_{s,\mathrm{red}}})$, while $\chi(\mathcal{O}_{Z_s}) = \chi(\mathcal{O}_{Z_{\mathrm{ggen}}}) = \chi(\mathcal{O}_{Y_{\mathrm{ggen}}}) = \chi(\mathcal{O}_{Y_s})$, since the generic fibre is unaltered by the collapsing. The final assertion follows by Proposition 2.2. □

The conditions for equality here may be inferred from the previous result and the fact that we then need $M = 1$ or $M = 2$ at each step of the blowing-down process.

PROPOSITION 2.5. *Assume that Y_{ggen} is connected, and that Y_s is non-reduced. Then $\chi(Y_s) > \chi(Y_{\mathrm{ggen}})$ unless each of Y_s and Y_{ggen} is a smooth elliptic curve.*

Proof. By Proposition 2.4, we have $\chi(\mathcal{O}_{Y_{s,\mathrm{red}}}) \geqslant \chi(\mathcal{O}_{Y_s})$, and this in turn is equal to $\chi(\mathcal{O}_{Y_{\mathrm{ggen}}})$. Hence

$$\begin{aligned}\chi(Y_{s,\mathrm{red}}) &= 2\chi(\mathcal{O}_{Y_{s,\mathrm{red}}}) + \mu(Y_{s,\mathrm{red}}) \\ &\geqslant 2\chi(\mathcal{O}_{Y_{\mathrm{ggen}}}) + \mu(Y_{s,\mathrm{red}}) = \chi(Y_{\mathrm{ggen}}) + \mu(Y_{s,\mathrm{red}}).\end{aligned}$$

We thus obtain $\chi(Y_s) = \chi(Y_{s,\mathrm{red}}) > \chi(Y_{\mathrm{ggen}})$ unless firstly, we have equality above, so Y_{ggen} is elliptic, and secondly, $\mu(Y_{s,\mathrm{red}}) = 0$, so $Y_{s,\mathrm{red}}$ is smooth. The result follows. □

According to **[25]**, this result is well known. We include the proof for completeness.

We conclude this section with a note about wild ramification in the case of non-isolated singularities. In view of Lemma 2.1 we can regard the ramification at C_i as wild when $p \mid m_i$, though the relation of this wildness condition to calculation of the Swan conductor is not clear.

LEMMA 2.6. *Suppose R_s is tame, that is, that no m_i is divisible by p. Then*

$$\mathrm{Sw}(V_s, s) = \sum_{x \in |Y_s|} (\mu_x^{\mathrm{red}}(\pi) - \mu_x(Y_{s,\mathrm{red}})).$$

Proof. It follows from our hypothesis that $R_s = Y_s - Y_{s,\mathrm{red}}$. Hence the expression for $D(s)$ given by (16) is equal to

$$K_Y . (Y_s - Y_{s,\mathrm{red}}) - (Y_s - Y_{s,\mathrm{red}})^2 + \sum_{x \in |Y_s|} \mu_x^{\mathrm{red}}(\pi).$$

Using (3) and (4), we may express the formula for $D(s)$ given by (12) as

$$\mu(Y_{s,\mathrm{red}}) - K_Y . Y_{s,\mathrm{red}} - Y_{s,\mathrm{red}}^2 + K_Y . Y_{\mathrm{ggen}} + Y_{\mathrm{ggen}}^2 + \mathrm{Sw}(V_s, s).$$

Comparing these two, and using $K_Y . Y_s = K_Y . Y_{\mathrm{ggen}}$ and $Y_s^2 = 0 = Y_s . Y_{s,\mathrm{red}} = Y_{\mathrm{ggen}}^2$, we see that the equation simplifies to the form stated. □

If R_s is not tame, and some component through x has multiplicity divisible by p, then the local equation has the form $f = g_1^p g_2$, and differentiating gives $\partial f = g_1^p \partial g_2$, so we expect no close relation between $\mu_x^{\mathrm{red}}(\pi)$ and $\mu_x(Y_{s,\mathrm{red}})$.

3. *Pencils of curves and Bertini's theorem*

Consider a smooth surface X_0 with a pencil of curves $\{\Gamma_t\}$ whose generic member Γ_{gen} is reduced and irreducible, but may be singular. If the field k has finite characteristic p, Bertini's theorem in its original form is not always valid. For example, for the plane pencil $t_0(x_0^3 + x_1^2 x_2) + t_1 x_2^3$ in characteristic 2, the curve Γ_t has a singular point at $(0 : u_0 : u_1)$ where $(u_0^2 : u_1^2) = (t_1 : t_0)$. A detailed study of systems of singular cubics appears in [**3**].

The correct formulation of Bertini's theorem for this case was found by Zariski in 1944 [**30**]: the introduction by Mumford to the first part of Zariski's collected works gives a useful account of Zariski's achievement in modern terminology.

The key is to consider the generic curve of the pencil as defined over the field $k(t)$. Since this field is not perfect, many of the familiar results of algebraic geometry do not hold without slight modifications. In particular, two of the traditional definitions of a 'simple point' on an algebraic variety are not here equivalent: if the variety is defined by equations whose differentials at that point cut out a tangent space of the same dimension as the variety, it is called *smooth* there; if the local ring of the variety at that point is regular, it is called a *regular* point.

Then Zariski's version of Bertini's theorem states that the generic curve Γ_{gen} of the pencil is regular outside the set of base points of the pencil. It is important here to distinguish between Γ_{gen} and Γ_{ggen}. Following the procedure in § 1.2, we study Γ_{gen} by blowing up all the base points of the pencil to obtain a pencil of curves Y_s on a surface Y having no base points. It follows from the theorem that Y_{gen} is regular (at all points).

It follows from elementary results about local rings that for curves the conditions 'regular' and 'normal' are equivalent. A function field of transcendence degree 1 over any field has an essentially unique normal model. The arguments leading to the proof of the Riemann–Roch theorem go through over any field (see, for example, Artin [**1**]). It follows from this theorem that for an irreducible normal curve C the genus and arithmetic genus agree, $g(C) = p_a(C)$.

The genus of a curve over a non-perfect field is not invariant under inseparable change of ground field (the first paper on this topic seems to be Tate [**29**], where it is shown that $g(C) - g(\overline{C})$ is a multiple of $\frac{1}{2}(p-1)$). (Since separable base change commutes with integral closure, and so preserves normality, this cannot affect the genus.) We may control the genus as follows. Since $\chi(\mathcal{O}_C)$ is constant in flat families (the constancy including both closed and non-closed points), it is invariant under change of ground field. On ground field extension however we may acquire non-regular points, and have to blow these up. If C_1 is the result of blowing up a closed point P on C, then taking Euler characteristics of the exact sequence

$$0 \to \mathcal{O}_C \to f_* \mathcal{O}_{C_1} \to \Big(\sum_{f(Q)=P} \mathcal{O}_1(Q) \Big) / \mathcal{O}_C(P) \to 0$$

we see that $\chi(\mathcal{O}_{C_1}) - \chi(\mathcal{O}_C)$ is the length of $\big(\sum_{f(Q)=P} \mathcal{O}_1(Q)\big) / \mathcal{O}_C(P)$; and this length is equal to $\frac{1}{2} m_P(m_P - 1)[K(P) : K]$ where m_P is the multiplicity of C at the point P and K is the field of definition of C (see Deligne [**8**] for a full treatment of this situation). The effect on the genus is thus to subtract the sum of the terms $\frac{1}{2} m_P(m_P - 1)[K(P) : K]$ corresponding to the points that have to be blown up. If we are extending the ground field to be algebraically closed, then all singular infinitely near points need to be resolved, and by (1) the sum is the double point number $\delta(\overline{C})$: thus for C regular, $g(\overline{C}) = g(C) - \delta(\overline{C})$. Alternatively we may argue that $g(\overline{C}) = p_a(\overline{C}) - \delta(\overline{C})$, while $p_a(\overline{C}) = p_a(C) = g(C)$.

Thus blowing up the base points of the pencil gives a regular generic fibre Y_{gen}, however Y_{ggen} may have singular points; necessarily only defined over inseparable extensions of $k(t)$. We have $g(Y_{\text{ggen}}) = g(Y_{\text{gen}}) - \delta(Y_{\text{ggen}})$. Artin uses the word *conservative* for the case when the genus is stable by base change: we see that this is equivalent to Y_{ggen} being non-singular. We may use the word *radical* for the opposite case.

If Γ_t has a singular point which moves with t, the pencil is radical. If $f_0 = ab_0^p + c_0h^2$ and $f_1 = ab_1^p + c_1h^2$ as polynomials on the plane P^2, then the general member $t_0f_0 + t_1f_1$ of the pencil is singular at points where $0 = t_0b_0^p + t_1b_1^p = h$. Thus any curve $h = 0$ may arise as a locus of singular points. It may be true that all cases may be put in this form. We can prove this if h is linear and have a sketch of proof whenever $h = 0$ is an irreducible rational curve.

Variable singular points may also arise on blowing up: consider an isolated base point of multiplicity m, with local coordinates (u, v) as usual. The first blow-up produces an exceptional curve with coordinate $(\xi : \eta)$: assume it is dicritical. The base points on this curve correspond to common zeros of f_m and g_m. The point $(1 : 0)$ is singular on the blow-up of $f = 0$ if v^2 divides f_m and v divides f_{m+1}. We have already seen that the exceptional curve is inseparable if f_m and g_m are both pth powers, or more generally if f_m/g_m is so; if in addition $g_mf_{m+1} = f_mg_{m+1}$ (for example, if $f_{m+1} = g_{m+1} = 0$), then each point of the exceptional curve is singular on some member of the pencil, which is thus radical.

The radical pencils were termed 'supercuspidal' and studied in [**27**], where some results of [**3**] were generalised. Observe that (since k is perfect) all elements of $k[t]$ become pth powers in $k[t^{1/p}]$, so all elements of $k(t)$ become pth powers in $k(t^{1/p})$. Thus if x is algebraic over $k(t)$ with minimal equation $f(x^{p^r}) = 0$ where f is separable, there is a separable g over $k(t^{p^{-r}})$ with $f(x^{p^r}) \cong g(x)^{p^r}$, so x is separable over $k(t^{p^{-r}})$. Thus any finite extension of $k(t)$ may be regarded as a separable extension of $k(t^{p^{-r}})$ for some r.

Consider the sequence $S_r \to S_{r-1} \to \ldots \to S_1 \to S_0 = S$ with each S_i isomorphic to P^1 and each map given by the Frobenius map $F(t) = t^p$. We can pull back the map π_0: $Y_0 \to S_0$ to give maps π_k: $Y_k \to S_k$. These are not given by pencils, but are certainly families of curves. If the field generated by all coordinates of all points blown up in the minimal resolution of Y_{ggen} is K, then by the above, K is separable over $k(t^{p^{-r}})$ for some r, so the normalisation of the generic fibre of Y_r is smooth.

Shimada's main result gives a normal form for the local behaviour at a generic point on the singular curve in the case $r = 1$.

THEOREM 3.1 [**27**]. *In suitable local coordinates in Y, if the above parameter r is* 1, *then π is given at a generic point on a singular curve by either*

(a) $\pi(x, y) = x^p - y^m$ *for some $m > 1$ prime to p, if Y_1 is separable over S_1;*

(b) $\pi(x, y) = x^p + y^q + x^{pm}y$ *for some $m > 0$, if the inseparable degree is q.*

We have not succeeded in obtaining a good insight into the structure of radical pencils. Perhaps the most natural construction is to resolve the singularities of the generic fibre. There are finitely many of these: one may make a base change, extending $k(t)$ to a field over which they (and all infinitely near points along which blowing up is required) are well defined, so that the field extension defines a finite map $S_1 \to S_0$. Note also that, geometrically, blowing up a point whose coordinates depend on t is perhaps best understood as taking the strict transform of $X_1 \subset X_0 \times S_0$ under the blow-up of a section of $X_0 \times S_0 \to S_0$. We can combine this with a base change by blowing up $X_0 \times S_0$ along the curve consisting of pairs (Q, t) such that Γ_t is singular at Q.

Eventually one must arrive at a smooth surface Y_1, a map $\pi_1: Y_1 \to S_1$, and a collection of curves E_r in Y_1 consisting of all the exceptional curves arising during the above procedure. We can call a point of S_1 or S_0 atypical if it is a critical value of the projection of any of the E_r, or the image of an intersection point of two distinct curves E_r. In some sense the other values should all be typical. But we see no way to control this sequence of operations: at the least, one would need a rather good understanding of equisingularity for the singularities deforming those of the generic fibre. We thus restrict from now on to the conservative case.

4. *Conservative pencils*

We saw in § 3 that conservative pencils are characterised by the property that if all the base points of the pencil are blown up, the generic fibre becomes smooth: this property is automatic in characteristic 0. We shall assume from now on that this holds and that, moreover, the generic fibre is irreducible: we may then use the results of § 2.

The point of main interest to us is to define what it means for a member of the pencil to be atypical, and to detect such members. For work over $\mathbb{C}$ we take 'atypical' to mean a member at which topological triviality of the family fails. We seek a model which we can follow for the finite characteristic case.

The following was proved in the complex case by Lê and Weber **[20]** by topological arguments (although that paper was addressed to the local case, the results are applicable globally).

THEOREM 4.1. (a) (4.3, loc. cit.) *The function $\mu(\Gamma_t)$ (interpreted as ∞ if Γ_t is non-reduced) is an upper semi-continuous function of t; and topological triviality holds along the set where it takes its minimum value.*

(b) *The complement of this set is finite, and consists of values where either Γ_t is non-reduced or $\chi(\Gamma_t) > \chi(\Gamma_{ggen})$: such values are called atypical.*

(c) (4.1, loc. cit.) *The value t is atypical if and only if either*

(C_1) *Y_t is singular,*

(C_2) *t is a branch value for some dicritical exceptional curve, or*

(C_3) *Y_t contains a point of intersection of two dicritical curves.*

(d) *The pencil is topologically trivial over the complement of the atypical set.*

We proceed to a direct combinatorial proof, valid in all characteristics, of a corresponding result.

THEOREM 4.2. *Let $\{\Gamma_t\}$ be a pencil of curves in a smooth surface X such that the generic curve Γ_{ggen} is irreducible. Then $\chi(\Gamma_t) \geqslant \chi(\Gamma_{\text{ggen}})$. Equality holds if and only if $Y_{t,\text{red}}$ is smooth,* (C$_2$) *and* (C$_3$) *both fail, and either Y_t is reduced or $Y_{t,\text{red}}$ (and hence also Y_{ggen}) is smooth elliptic.*

We begin the proof by finding a formula expressing the difference $\chi(\Gamma_t) - \chi(\Gamma_{\text{ggen}})$ as a sum of terms which we will then prove to be non-negative. A consideration of the cases when they vanish will lead to the conclusion. As in § 1.2 we blow up all base points of the pencil to give a map $\pi: Y \to S$.

We need to take care with notation. Write Y_t for the special fibre. As a divisor, we can write this as a sum $\widetilde{\Gamma}_t + E_t$, where E_t is the sum of terms corresponding to exceptional curves (for the map $Y \to X$) lying in the fibre Y_t. We add a further suffix 'red' to refer to the underlying reduced curves. We write D for the union of all dicritical curves D_i in Y, and define the $\mathbb{Q}$-divisor D_Q to be the sum over i of terms $p^{-a_i}[D_i]$, where p^{a_i} is the degree of inseparability of the projection $D_i \to S$.

LEMMA 4.3. *We have $\chi(\Gamma_t) - \chi(\Gamma_{\text{ggen}}) = \sum_1^6 N_r$, where*

$$N_1 = 2(\chi(\mathcal{O}_{Y_{t,\text{red}}}) - \chi(\mathcal{O}_{Y_t})), \qquad N_2 = \mu(\widetilde{\Gamma}_{t,\text{red}}),$$
$$N_3 = \widetilde{\Gamma}_{t,\text{red}} \cdot E_{t,\text{red}} - \#(\widetilde{\Gamma}_t \cap E_t), \quad N_4 = \widetilde{\Gamma}_{t,\text{red}} \cdot E_{t,\text{red}} - \chi(\mathcal{O}_{E_{t,\text{red}}}),$$
$$N_5 = \#(D \cap E_t) - \chi(\mathcal{O}_{E_{t,\text{red}}}), \qquad N_6 = Y_t \cdot D_Q - \#(Y_t \cap D).$$

Proof. By (5), $\chi(\mathcal{O}_{Y_{t,\text{red}}}) = \chi(\mathcal{O}_{\widetilde{\Gamma}_{t,\text{red}}}) + \chi(\mathcal{O}_{E_{t,\text{red}}}) - \widetilde{\Gamma}_{t,\text{red}} \cdot E_{t,\text{red}}$. Since also $\chi(\mathcal{O}_{Y_{\text{ggen}}}) = \chi(\mathcal{O}_{Y_t})$,

$$\chi(\mathcal{O}_{\widetilde{\Gamma}_{t,\text{red}}}) - \chi(\mathcal{O}_{Y_{\text{ggen}}}) = (\chi(\mathcal{O}_{Y_{t,\text{red}}}) - \chi(\mathcal{O}_{Y_t})) - \chi(\mathcal{O}_{E_{t,\text{red}}}) + \widetilde{\Gamma}_{t,\text{red}} \cdot E_{t,\text{red}}. \tag{17}$$

Now $\chi(\widetilde{\Gamma}_{t,\text{red}}) = 2\chi(\mathcal{O}_{\widetilde{\Gamma}_{t,\text{red}}}) + \mu(\widetilde{\Gamma}_{t,\text{red}})$, while the generic fibre is smooth, so

$$\chi(\widetilde{\Gamma}_{t,\text{red}}) - \chi(Y_{\text{ggen}}) = 2(\chi(\mathcal{O}_{\widetilde{\Gamma}_{t,\text{red}}}) - \chi(\mathcal{O}_{Y_{\text{ggen}}})) + \mu(\widetilde{\Gamma}_{t,\text{red}}). \tag{18}$$

On the other hand, the difference $\chi(\widetilde{\Gamma}_t) - \chi(\Gamma_t)$ can be evaluated by counting the numbers of points in preimages under $\widetilde{\Gamma}_t \to \Gamma_t$. This map is bijective outside the set B of base points of the pencil. We must thus count the number of points of $\widetilde{\Gamma}_t$ lying over the points of B. These are the points of $\widetilde{\Gamma}_t$ which also lie on one or more exceptional curves. These curves may be dicritical or contained in the fibre Y_t, and hence in E_t. Thus we have

$$\chi(\widetilde{\Gamma}_t) - \chi(\Gamma_t) = \#(\widetilde{\Gamma}_t \cap (D \cup E_t)) - \#B.$$

Here we expand

$$\#(\widetilde{\Gamma}_t \cap (D \cup E_t)) = \#(\widetilde{\Gamma}_t \cap D) + \#(\widetilde{\Gamma}_t \cap E_t) - \#(\widetilde{\Gamma}_t \cap D \cap E_t).$$

We proceed similarly with the generic fibre, save that E_{gen} is empty. Comparing these shows that $\chi(\Gamma_t) - \chi(\Gamma_{\text{ggen}})$ is equal to

$$(\chi(\widetilde{\Gamma}_t) - \chi(Y_{\text{ggen}})) + (\#(Y_{\text{ggen}} \cap D) - \#(\widetilde{\Gamma}_t \cap D)) - \#(\widetilde{\Gamma}_t \cap E_t) + \#(\widetilde{\Gamma}_t \cap D \cap E_t). \tag{19}$$

The definition of D_Q was so framed that for each dicritical curve D_i, its intersection number with the fibre at a generic point is p^{a_i} so the intersection number $D_Q . Y_{\text{ggen}}$ is 1. Hence $D_Q . Y_{\text{ggen}} = \#(D \cap Y_{\text{ggen}})$. Since Y_t is linearly equivalent to Y_{ggen}, we have $Y_{\text{ggen}} . D = Y_t . \mathrm{D}$ and so

$$\#(Y_{\text{ggen}} \cap D) = Y_{\text{ggen}} . D_Q = Y_t . D_Q. \tag{20}$$

Substituting the right-hand side of (17) for $\chi(\mathcal{O}_{\widetilde{\Gamma}_{t,\text{red}}}) - \chi(\mathcal{O}_{Y_{\text{ggen}}})$ in (18), and substituting the result and (20) in (19) gives an expression for $\chi(\Gamma_t) - \chi(\Gamma_{\text{ggen}})$ which is the sum of N_1, N_2, N_3, N_4 and

$$-\chi(\mathcal{O}_{E_{t,\text{red}}}) - \#(\widetilde{\Gamma}_t \cap D) + \#(\widetilde{\Gamma}_t \cap D \cap E_t) + Y_t . D_Q.$$

But since $\#(D \cap Y_t) = \#(D \cap \widetilde{\Gamma}_t) + \#(D \cap E_t) - \#(D \cap \widetilde{\Gamma}_t \cap E_t)$, this reduces to $N_5 + N_6$. □

We next show that each of the terms $N_1, \ldots, N_6$ is non-negative. Indeed, the assertion for N_1 was proved in Proposition 2.4. Non-negativity of N_2 and N_3 is immediate, and follows for N_6 since at each intersection point of Y_t with any D_i, the intersection number is the local degree of the projection of D_i on S, which is a multiple of p^{a_i}.

Since the exceptional curves are constructed by a sequence of blowings-up of points in a smooth surface, they (or rather, the dual graph) form a forest with normal crossings. In particular, the curves E_t contained in Y_t form a forest, which is a union of trees T. Each tree T_{red} consists of smooth rational curves with normal crossings, thus each having $\chi(\mathcal{O}) = 1$, and each edge of the dual graph corresponds to an intersection number 1. Since the number of vertices exceeds that of edges by 1, it follows from (5) that $\chi(\mathcal{O}_{T_{\text{red}}}) = 1$.

As Y_t is connected, and T is not the whole fibre, it must meet the union of the others; hence $T \cap \widetilde{\Gamma}_t$ is non-empty. Thus $T_{\text{red}} . \widetilde{\Gamma}_{t,\text{red}} - \chi(\mathcal{O}_{T_{\text{red}}}) \geqslant 0$, and non-negativity of N_4 follows by summing over trees T. We also see that non-negativity of N_5 will follow if each tree T intersects D: we address this next.

We recall from § 1.1 the notion 'a_β lies above a_α', which gives a partial order on the set of infinitely near points; we say that a set J of infinitely near points is *convex* if $a_\alpha < a_\beta < a_\gamma$ and $a_\alpha \in J$, $a_\gamma \in J$ imply $a_\beta \in J$.

LEMMA 4.4. *For each T, every maximal component E of T meets D. Hence $T \cap D \neq \emptyset$.*

Proof. Let E_m be a component of T which is maximal in the above partial order. Since e_m is a base point, in the surface X_m where E_m has just been created E_m meets Y_{ggen}, in P, say. This cannot lie on any other component of T since otherwise blowing it up would disconnect T which, by hypothesis, remains connected in Y.

Consider the sequence of blowings-up at points of E_m over P in higher surfaces in the sequence until the corresponding point ceases to lie on Y_{ggen}. This produces a chain of exceptional curves $A_1, A_2, \ldots, A_k$ with each intersecting E_m at P at its first appearance, so that no further blowing up takes place over P. Since T is a connected component, A_k cannot lie in the fibre Y_t: nor can it lie in another fibre, since this is disjoint from Y_t. Hence A_k is dicritical, and the result follows. □

This completes the proof of the inequality in Theorem 4.2. We next seek to characterise the cases of equality.

LEMMA 4.5. *Suppose equality holds in Theorem 4.2. Then the following hold.*

(i) *Each component of* $\widetilde{Y}_{t,\mathrm{red}}$ *is smooth, and* $\widetilde{Y}_{t,\mathrm{red}}$ *has normal crossings. Each component* T *of* E_t *meets* $\widetilde{\Gamma}_t$ *in just one point,* $\widetilde{\Gamma}_t$ *has just one component, and the dual graph of* Y_t *is a tree.*

(ii) *If* Y_t *is not reduced, it has just one component which is smooth, elliptic, and of multiplicity a power of* p.

(iii) *Neither* C_2 *nor* C_3 *holds. Each tree* T *meets* D *in just one point.*

(iv) *For each* T, *the set of components of* T *is totally ordered and convex.*

Proof. (i) As $N_2 = 0$, $\widetilde{\Gamma}_{t,\mathrm{red}}$ is smooth (and components of $E_{t,\mathrm{red}}$ are smooth anyway); as $N_4 = 0$, it follows from the above discussion that for each component T of E_t we have $T_{\mathrm{red}} \cdot \widetilde{\Gamma}_{t,\mathrm{red}} = 1$, so T meets $\widetilde{\Gamma}_t$ in just one point, transversely. Thus $Y_{t,\mathrm{red}}$ has normal crossings. Since Y_t is connected, it follows that $\widetilde{\Gamma}_t$ is connected, so has just one component, and the dual graph of Y_t is a tree.

(ii) Suppose Y_t is non-reduced. Since $N_1 = 0$, it follows from Proposition 2.4 that if all exceptional curves of the first kind in Y_t are collapsed in turn we obtain a fibre of Kodaira type mI_b for some $m \geqslant 2$ and $b \geqslant 0$. If $b \geqslant 1$, the dual graph is not a tree, but contains a cycle (of length b). Blowing up points to produce exceptional curves will still leave a cycle (perhaps of greater length). Hence this case does not occur.

Since $b = 0$, there is a component of Z_t, and hence of Y_t, of genus 1: this component must be $\widetilde{\Gamma}_t$. Since by Lemma 1.2 no other component may be exceptional of the first kind, Y_t just has one component C. Finally, if C has multiplicity m as component of the fibre, and D is dicritical of inseparable degree p^i, the local intersection number at any point $P \in C \cap D$ satisfies $p^i = D \cdot Y_t = mD \cdot C$, so m is a power of p.

(iii) Next we show that $N_6 > 0$ if and only if either C_2 or C_3 holds. Consider a point P of intersection of Y_t with a dicritical D_i. The intersection $D_i \cdot Y_t$ is p^{a_i} if we have no ramification at t, otherwise it is greater. Thus if C_2 holds, the point P contributes at least 1 to $Y_t \cdot D_Q$, and this is also true if C_3 holds and we have two dicritical curves through P. The converse follows by the same argument.

The vanishing of N_5 shows that each T meets D in just one point.

(iv) Since the partial order on the set of exceptional curves gives this the structure of a 1-way tree, if T is not a chain it has more than one maximal element. But the proof of Lemma 4.4 shows that each maximal element meets D, so then $\#(T \cap D) \geqslant 2$, contradicting the above.

Let $\overline{T}$ denote the convex cover of T, that is, the set of all exceptional curves C such that for some $A, B \in T$ we have $A < C < B$. Since each blow-up produces a component intersecting the union of the preceding ones, the union of all curves in $\overline{T}$ is connected. Thus if it includes any components not in T, at least one of these – necessarily dicritical – will meet T. This intersection of T with D is distinct from the one constructed in Lemma 4.4 since this dicritical precedes E_m while the other follows it. Thus again we have a contradiction. □

Now suppose that $Y_{t,\mathrm{red}}$ is smooth and C_2 and C_3 fail. Since $Y_{t,\mathrm{red}}$ is smooth, it is irreducible and $\widetilde{\Gamma}_{t,\mathrm{red}}$ is smooth. Thus $N_2 = 0$ and E_t is empty, so N_3, N_4 and N_5 all vanish. As we have just seen, since C_2 and C_3 fail, $N_6 = 0$. Now N_1 vanishes if Y_t is reduced, and also if $Y_{t,\mathrm{red}}$ is smooth elliptic, so in these cases we have $\chi(\Gamma_t) = \chi(\Gamma_{\mathrm{ggen}})$.

To complete the proof of Theorem 4.2, it remains to show that in the case of equality $Y_{t,\mathrm{red}}$ is smooth, that is, E_t is empty. We will assume this is false, and aim for a contradiction.

LEMMA 4.6. *Under the conditions of Lemma* 4.5, $E_t = \emptyset$.

Proof. Suppose not; then by Lemma 4.5, Y_t is reduced. Choose a component T of E_t such that no exceptional curve in E_t lies above the highest curve V_m in T (if there is more than one highest curve in T, then by Lemma 4.4, $\#(T \cap D) \geqslant 2$). Write $Y_t = T + W_t$.

Since (by Lemma 4.5) T is convex, in the chain of blowings-up that produces Y, the components $V_1, \ldots, V_m$ of T appear consecutively, so we have surfaces $X_0, \ldots, X_m$ say. There is just one base point P, say in T in X_m, and it lies in V_m (this follows since $\#(T \cap D) = 1$). Resolving this until there is no base point on V_m gives the chain $D_1, D_2, \ldots, D_k$ of curves constructed in the proof of Lemma 4.4 (all now dicritical since V_m is the highest vertical curve). If $Z_0 = X_m$, these appear in turn on surfaces $Z_1, \ldots, Z_k$. Any further blowing up to produce Y takes place at points not on T and produces further dicritical curves.

We will use the same letter to denote the image of a curve in Y in any of these intermediate surfaces, but avoid ambiguity by writing $(A\,.\,B)_Z$ to denote the intersection number of A and B considered as curves in the surface Z. Also if a blow-up at a point in one surface produces an exceptional curve in the next, we denote the point by the same letter as the curve, but in lower case, for example, $D_i \to d_i$, and then $m_{d_i}(A)$ denotes the multiplicity of the curve A at the point d_i in the lower surface. Recall from § 1.1 that if A and B are curves through a smooth point e on a surface X, and blowing e up produces an exceptional curve E and strict transforms A and B in Z, then $m_e(A) = (E\,.\,A)_Z$ and $(A\,.\,B)_X = m_e(A)m_e(B) + (A\,.\,B)_Z$.

Blowing up v_1 produces a vertical curve. Hence the multiplicity at v_1 of the member Y_t of the pencil strictly exceeds the multiplicity of Y_{ggen}. We shall obtain a contradiction to this inequality.

We have $m_{d_i}(Y_{\mathrm{ggen}}) - m_{d_i}(W_t) = (D_i\,.\,(Y_{\mathrm{ggen}} - W_t))_{Z_i}$. As Y_{ggen} and Y_t are linearly equivalent, and $Y_t - W_t = T$, this equals $(D_i\,.\,T)_{Z_i}$. But the only component of T meeting D_i in Z_i is V_m, and as V_m is smooth in Z_{i-1} this intersection number is 1.

Since the only intersections of V_m with Y_{ggen} or W_t in Z_{i-1} occur at d_i, we have

$$(V_m\,.\,Y_{\mathrm{ggen}} - V_m\,.\,W_t)_{Z_{i-1}} = m_{d_i}(Y_{\mathrm{ggen}}) - m_{d_i}(W_t) + (V_m\,.\,Y_{\mathrm{ggen}} - V_m\,.\,W_t)_{Z_i},$$

and so adding up,

$$(V_m\,.\,Y_{\mathrm{ggen}} - V_m\,.\,W_t)_{Z_0} = k + (V_m\,.\,Y_{\mathrm{ggen}} - V_m\,.\,W_t)_{Z_k}.$$

But in Z_k, V_m is disjoint from Y_{ggen} and meets W_t transversely in just one point. Thus the right-hand side here is equal to $k - 1$. The left-hand side is equal (by the same argument) to $V_m\,.\,(Y_t - W_t) = V_m\,.\,T$. This is equal to $r - 1$ where r is the number of other components of T meeting V_m. In particular, if T has just one component V_m, we obtain $0 = r = k$, contradicting the fact that there is at least one dicritical meeting V_m.

Until V_m has been blown down, none of the curves not in T meet V_{m-1}, so none of the corresponding infinitely near points are proximate to v_{m-1}. It follows from (2) that each component of Y_{ggen} and W_t has the same multiplicity at v_{m-1} as at v_m, and hence that

$$m_{v_{m-1}}(Y_{\text{ggen}}) - m_{v_{m-1}}(W_t) = m_{v_m}(Y_{\text{ggen}}) - m_{v_m}(W_t) = k - 1 \geqslant 0. \tag{21}$$

In $X_m = Z_0$ none of the components of T other than V_m meet any component of Y_{ggen} or W_t outside T. Thus any such component which passes through v_1 must also pass through v_m, and no further infinitely near point of either Y_{ggen} or W_t is proximate to v_{m-1}. We may thus apply Lemma 1.1 to see that there are non-negative numbers a and b, determined solely by the proximity relations, such that for any component A of these curves through v_m, $m_{v_1}(A) = am_{v_{m-1}}(A) + bm_{v_m}(A)$.

We apply this to Y_{ggen} on one hand and to W_t on the other. It then follows from (21) that $m_{v_1}(Y_{\text{ggen}}) - m_{v_1}(W_t) = (a+b)(k-1) \geqslant 0$. However, since in X_0 all components of T have been blown down, $m_{v_1}(Y_t) = m_{v_1}(W_t)$. Hence we have $m_{v_1}(Y_{\text{ggen}}) \geqslant m_{v_1}(Y_t)$, contradicting the fact that V_1 is vertical. This contradiction establishes the result. □

We have proved a little more than asserted in Theorem 4.2: we see from Lemma 4.5 that if equality holds and $Y_t = mC$ is non-reduced (so C is smooth elliptic), and if also the pencil has base points and hence Y contains dicritical curves, the degree of inseparability of each of these must be divisible by m.

References

1. E. Artin, *Algebraic numbers and algebraic functions* (Gordon and Breach, New York, 1967).
2. S. J. Bloch, 'Cycles on arithmetic schemes and Euler characteristics of curves', *Algebraic geometry – Bowdoin 1985* (ed. S. J. Bloch *et al.*), Proceedings of Symposia in Pure Mathematics 46, Part II (American Mathematical Society, Providence, RI, 1987) 421–450.
3. E. Bombieri and D. Mumford, 'Enriques' classification of surfaces in characteristic p III', *Invent. Math.* 35 (1976) 197–232.
4. S. A. Broughton, 'On the topology of polynomial hypersurfaces', *Singularities* (ed. P. Orlik), Proceedings of Symposia in Pure Mathematics 40, Part I (American Mathematical Society, Providence, RI, 1983) 167–178.
5. S. A. Broughton, 'Milnor numbers and the topology of polynomial hypersurfaces', *Invent. Math.* 92 (1988) 217–241.
6. A. Campillo, *Algebroid curves in positive characteristic*, Lecture Notes in Mathematics 813 (Springer, Berlin, 1980).
7. P. Deligne *et al.*, *Groupes de monodromie en géométrie algébrique, II, Séminaire de Géometrie Algébrique du Bois-Marie 1967–69* (SGA 7 II), Lecture Notes in Mathematics 340 (Springer, Berlin, 1973).
8. P. Deligne, 'Intersections sur les surfaces régulières', [**7**, exposé X].
9. P. Deligne, 'Le formalisme des cycles évanescents', [**7**, exposé XIII].
10. P. Deligne, 'La formule de Milnor', [**7**, exposé XVI].
11. I. Dolgachev, 'On the purity of the degeneration loci of families of curves', *Invent. Math.* 8 (1969) 34–54.
12. I. Dolgachev, 'The Euler characteristic of a family of algebraic varieties', *Math. USSR Sb.* 18ii (1972) 303–319.
13. W. Fulton, *Intersection theory* (Springer, Berlin, 1984).
14. A. Grothendieck *et al.*, *Cohomologie l-adique et fonctions L, Séminaire de Géometrie Algébrique du Bois-Marie 1965–66* (SGA 5), Lecture Notes in Mathematics 589 (Springer, Berlin, 1977).
15. A. Grothendieck, 'Formule d'Euler–Poincaré en cohomologie étale', [**14**, exposé X].

16. HÀ HUY VUI and LÊ DŨNG TRÁNG, 'Sur la topologie des polynômes complexes', *Acta Math. Vietnam.* 9 (1984) 21–32.
17. R. HARTSHORNE, *Algebraic geometry*, Graduate Texts in Mathematics 52 (Springer, Berlin, 1977).
18. B. IVERSEN, 'Critical points of an algebraic function', *Invent. Math.* 12 (1971) 210–224.
19. K. KODAIRA, 'On compact analytic surfaces II', *Ann. of Math.* 77 (1963) 563–626.
20. LÊ DŨNG TRÁNG and CLAUDE WEBER, 'Equisingularité dans les pinceaux de germes de courbes planes et C^0-suffisance', *Enseign. Math.* 43 (1997) 355–380.
21. A. P. OGG, 'Cohomology of abelian varieties over function fields', *Ann. of Math.* 76 (1962) 185–212.
22. A. P. OGG, 'Elliptic curves and wild ramification', *Amer. J. Math.* 89 (1967) 1–21.
23. M. RAYNAUD, 'Caractéristique d'Euler–Poincaré d'un faisceau et cohomologie des variétés abéliennes', *Dix exposés sur la cohomologie des schémas* (ed. J. Giraud, North-Holland, Amsterdam, 1968) 12–30.
24. I. R. ŠAFAREVIČ, 'Principal homogeneous spaces defined over a function field', *Amer. Math. Soc. Transl.* (2) 37 (1964) 85–114.
25. T. SAITO, 'Conductor, discriminant and the Noether formula of arithmetic surfaces', *Duke Math. J.* 57 (1988) 151–173.
26. J.-P. SERRE, *Représentations des groupes finis*, 2nd edn (Hermann, Paris, 1971).
27. I. SHIMADA, 'On supercuspidal families of curves on a surface in positive characteristic', *Math. Ann.* 292 (1992) 645–669.
28. X. SUN, 'Ramifications on arithmetic schemes', *J. reine angew. Math.* 488 (1977) 37–54.
29. J. TATE, 'Genus change in inseparable extensions of base fields', *Proc. Amer. Math. Soc.* 3 (1952) 400–406.
30. O. ZARISKI, 'The theorem of Bertini on the variable singular points of a linear system of varieties', *Trans. Amer. Math. Soc.* 56 (1944) 130–140; *Oscar Zariski: Collected Papers*, Vol. I (ed. H. Hironaka and D. Mumford, MIT Press, Cambridge, MA, 1972) 242–252.

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